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Turbulent Mixing Noise from Supersonic Jets

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There is now a substantial body of theoretical and experimental evidence that the dominant part of the turbulent mixing noise of supersonic jets is generated directly by the large turbulence structures/instability waves of the jet flow. Earlier, Tam and Burton provided a description of the physical mechanism by which supersonically traveling instability waves can generate sound efficiently. They used the method of matched asymptotic expansions to construct an instability wave solution which is valid in the far field. The present work is an extension of the theory of Tam and Burton. It is argued that the instability wave spectrum of the jet may be regarded as generated by stochastic white noise excitation at the nozzle lip region. The reason why the excitation has white noise characteristics is that near the nozzle lip region the flow in the jet mixing layer has no intrinsic length and time scales. The present stochastic wave model theory of supersonic jet noise contains a single unknown multiplicative constant. Comparisons between the calculated noise directivities at selected Strouhal numbers and experimental measurements of a Mach 2 jet at different jet temperatures have been carried out. Favorable agreements are found.

I. Introduction

SUPERSONIC jet noise consists of three main components. They are the turbulent mixing noise, the broadband shock associated noise, and the screech tones. The latter two noise components are generated only when the jet is imperfectly expanded and a shock cell structure is formed in the jet plume. This paper considers perfectly expanded jets and turbulent mixing noise alone.

There is now a substantial body of theoretical and experimental evidence that the dominant part of the turbulent mixing noise from supersonic jets is generated directly by the large turbulence structure/instability waves of the jet flow (see Refs. 1 and 2). Figure 1 shows the directivities of turbulent mixing noise from an isothermal Mach 2 jet (total temperature = 500 K) at four selected Strouhal numbers measured by Seiner et al.[3] The data indicates that for inlet angles less than 110 deg, the jet noise radiation is almost uniform without any preferred direction. It is believed that this low level, almost uniform background noise is generated by the fine-scale turbulence of the jet flow. The predominant part of the turbulent mixing noise is radiated in the downstream directions at inlet angles greater than 130 deg. This part of the noise is generated directly by the large turbulence structures/instability waves of the jet flow. The prediction of this dominant part of turbulent mixing noise is the primary objective of this paper. The physical mechanism by which the large turbulence structures/instability waves generate sound was described by Tam and Burton.[1] The noise generation process is most efficient when the propagation velocity of the instability wave is supersonic relative to the ambient speed of sound. It is well known in gas dynamics that supersonic flow past a wavy wall can cause intense Mach wave radiation. A supersonic instability wave, in many ways, is analogous to a supersonically traveling wavy wall. Thus, such an instability wave is an efficient noise generator.

In the mixing layer of a supersonic jet, the flow and all of the turbulence statistics exhibit self-similarity. The spreading rate of the jet is very small so that the turbulence characteristics change very slowly in the downstream direction. These suggest that the turbulent motion of the flow is locally in a state of quasiequilibrium. According to standard statistical mechanics theory, when a system is in a state of equilibrium, the large-scale fluctuations of the system can be represented by a linear superposition of its normal modes. For the jet flow, a good representation of the large-scale turbulent motion in a statistical sense, therefore, is to use a linear superposition of the normal modes of the jet flow. The important normal modes in this case are the instability wave modes of the jet. Thus, from this point of view, the large-scale turbulence structures of the jet and the instability wave modes are synonymous. The latter could be regarded as a mathematical representation of the former. Previously, Tam and Chen[4] have made use of this reasoning to develop a turbulence theory of two-dimensional mixing layers. Subsequently, Plaschko[5,6] applied the theory to calculate certain statistical properties of turbulent jets with success. In this work, we will adopt this statistical point of view concerning jet turbulence, i.e., we will represent the large-scale turbulent motion of a supersonic jet by a linear combination of all of the instability wave modes of the jet. The amplitude of the instability waves are, however, treated as stochastic random functions in accordance with the indeterministic nature of jet turbulence.

Classical instability wave theory is formulated as an eigenvalue problem. To ensure that the boundary condition is homogeneous the theory requires the instability wave solution to decay to zero (exponentially) away from the jet. Because of this requirement classical instability wave theory can never predict acoustic radiation. This problem was first recognized and studied by Tam and Morris.[7] They showed that the classical solution based on the lo-

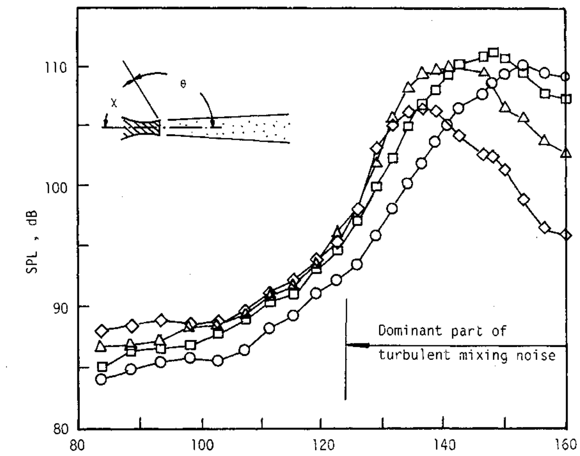

Fig. 1 Measured[3] noise directivities at selected Strouhal numbers *St* of a Mach 2 jet at a total temperature of 500 K: ○ *St* = 0.067, □ *St* = 0.12, △ *St* = 0.20, and ◊ *St* = 0.40.

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cally parallel flow approximation is but the first term of a multiple-scales expansion of the solution of a more rigorously formulated instability wave theory. The multiple-scales expansion is not uniformly valid. It is a proper asymptotic expansion when applied to points inside the jet. For the space outside, the expansion is invalid and should not be used.

Subsequently, Tam and Burton[1] developed a uniformly valid instability wave solution for axisymmetric supersonic jets by the method of matched asymptotic expansions. In their approach, the physical domain is divided into two parts. The first region consists of the jet flow and the near field surrounding the jet. The second region comprises all of the physical space outside the jet. It is assumed that there is an overlap between the two regions. Physically, the necessity to divide the entire space into two regions is quite obvious. Inside the jet the mean flow changes very slowly in the axial direction as compared to the radial direction. This large disparity in the spatial rate of change in the two directions provides a natural small parameter for a multiple-scales expansion. Outside the jet, particularly in the acoustic field, there is no longer any preferred direction. Multiple-scales solution would, therefore, not be appropriate. In applying the method of matched asymptotic expansions to the instability wave problem, two independent asymptotic expansions of the solution of the linearized equations of motion of a compressible inviscid flow are constructed using the slow spreading rate of the jet as the small parameter. The inner solution, in the form of a multiple-scales asymptotic expansion, is valid inside and in the adjacent region outside the jet, i.e., the first region. The outer solution or expansion, constructed by the method of Fourier transform, is valid outside the jet all of the way to the acoustic far field, i.e., the second region. To guarantee that the two asymptotic expansions are expansions of the same solution but valid over different parts of the physical space, they are matched in the overlapping region according to the intermediate matching principle.[8,9] Matching of the expansions to the lowest order terms gives the eigenvalue of the classical instability wave problem. Matching to the first-order terms gives the nonparallel flow correction. It also removes the arbitrariness inherent in the normalization of the local eigenfunction. In this way, Tam and Burton were able to construct a uniformly valid instability wave solution. To determine the pressure and velocity fluctuations associated with the instability wave inside the jet, the inner solution is used. To determine the acoustic near field outside the jet and the radiated noise in the far field, the outer solution is used. Numerical results of the near field sound-pressure-level contours at Strouhal numbers 0.2 and 0.4 for a Mach 2.1 cold jet (total temperature of jet is equal to ambient temperature) obtained by Tam and Burton compared well with the experimental measurements of Troutt and McLaughlin.[10] There were also favorable agreements between the calculated and measured jet centerline mass-velocity fluctuations associated with the instability waves.

Recently, Tam et al.[11] argued that if, indeed, large-scale instability waves were the dominant sources of high-speed jet noise then the frequency at the peak of the noise spectrum must be nearly equal to the frequency of the most amplified instability wave of the jet. Furthermore, the direction of peak noise radiation must also be nearly equal to the direction of Mach wave radiation of the most amplified instability wave. They compared the calculated frequencies and peak noise directions with the hot jet noise measurements of Tanna et al.[12] Three sets of noise spectra measured at fully expanded jet Mach numbers of 1.4, 1.7, and 2.0 were used. The jet to ambient temperature ratio covers the range from cold jet to a ratio of 2.5. Good agreements were found especially at high jet temperatures. (High jet temperature means high jet velocity. This ensures that the instability waves are propagating at highly supersonic velocities relative to ambient sound speed. Thus, they become the dominant source of noise even for low supersonic Mach number jets.) Most recently, McLaughlin et al.[13] used helium/air supersonic jets at low Reynolds number to study the relationship between the instability waves of the jet and noise radiation. Their measured Strouhal numbers of the most amplified jet instability waves agreed well with the calculated results of Tam et al.[11] At low Reynolds number, not surprisingly, the radiated noise is dominated by tones. These tones have the same frequencies as the most amplified instability waves.

The work of Tam and Burton[1] concentrates on a single instability wave. In a high Reynolds number fully turbulent supersonic jet, there is a wide spectrum of instability waves. To be able to predict supersonic jet noise, the entire instability wave spectrum must be considered. The purpose of this work is to make use of the single instability wave solution of Tam and Burton to construct a broadband jet noise theory. A stochastic model of the instability waves will be developed. This model allows the relative amplitudes of the instability waves of different frequencies and azimuthal modes to be determined. With this information, the dominant part of the noise spectrum and directivity can be predicted from first principle up to a single multiplicative constant.

Seiner et al.[3] recently obtained extensive narrow-band noise measurements for a Mach 2.0 supersonic jet at different jet temperatures. Numerical results of the directivity of such a jet at selected Strouhal numbers will be presented in Sec. III. On comparing the calculated directivities with measurements, good agreements are found. This provides strong support for the present stochastic model theory.

II. Stochastic Instability Wave Model of the Large Turbulence Structures and Noise of Supersonic Jets

Consider small amplitude disturbances superimposed on the mean flow of an inviscid perfectly expanded supersonic jet. Dimensionless variables with radius of the jet at nozzle exit R_j, jet exit velocity u_j, R_j/u_j, jet density at nozzle exit ρ_j, and $\rho_j u_j^2$ as the length, velocity, time, density and pressure scales will be used throughout the analysis. The Mach number of the jet and the (dimensionless) ambient gas density will be denoted by M_j and ρ_∞, respectively. We will begin with the single instability wave solution of Ref. 1. The formulas for the pressure fluctuation p will be written out subsequently. Formulas for the other variables are similar. Dependence on ε, the perturbation parameter in the matched asymptotic solution of Ref. 1, will not be displayed. With respect to a cylindrical coordinate system (r, ϕ, x) centered at the nozzle exit, to the lowest order, the inner solution for an instability wave of angular frequency ω and azimuthal mode number n has the form

$$p_{\text{inner}}(r, x, \phi, t; \omega, n) = A_0(x, n, \omega)\hat{p}_n(r, x, \omega) \exp\{i[\theta_n + n\phi - \omega t + (\pi/2)]\} \tag{1}$$

where $\theta_n(x, \omega) = \int_0^x \alpha_n(x, \omega)\,dx$; and $\alpha_n(x, \omega)$ is the local wavenumber (eigenvalue of the instability wave problem). $\hat{p}_n(r, x, \omega)$ is the eigenfunction, normalized so that

$$\hat{p}_n(r, x, \omega) \to H_n^{(1)}[i\lambda(\alpha, \omega)r] \tag{2}$$

as $r \to \infty$. $H_n^{(1)}[\ ]$ is the nth order Hankel function of the first kind. The function $\lambda(\alpha, \omega)$ is given by

$$\lambda(\alpha, \omega) = (\alpha^2 - \rho_\infty M_j^2 \omega^2)^{1/2}$$

The branch cuts of the preceding square root function are to be taken so that $-\pi/2 \le \arg\lambda \le \pi/2$. The left (right) equality sign is to be removed if ω is negative (positive). In Eq. (1), the function $A_0(x, n, \omega)$ is the amplitude function of the instability wave. It is related to the nonparallel flow correction factor $\beta_n(x, \omega)$ as

$$A_0(x, n, \omega) = \hat{A}_n(\omega)e^{-\beta_n(x,\omega)}$$
$$\beta_n(x, \omega) = \int_0^x \frac{I_2}{I_1}\,dx \tag{3}$$

The detailed formulas for I_1 and I_2 can be found in Ref. 1. The wave amplitude at the nozzle exit $\hat{A}_n(\omega)$ is arbitrary within the framework of the single instability wave theory. In the present stochastic wave model theory, the large turbulence structures of the jet flow are represented by a linear superposition of instability waves of all frequencies and azimuthal mode numbers. The large turbulence structures are random and not completely deterministic. To simulate this characteristic, the initial wave amplitude $\hat{A}_n(\omega)$ will be taken as a stochastic random function. For convenience, the normalized initial amplitude $a_n(\omega)$ will be used instead. Normalized amplitude $a_n(\omega)$ is related to $\hat{A}_n(\omega)$ by

$$\hat{A}_n(\omega) = \frac{a_n(\omega)}{|\hat{p}_n(r_{1/2}, 0, \omega)|} \tag{4}$$

where $|\hat{p}_n(r_{1/2}, 0, \omega)|$ is the magnitude of the pressure eigenfunction at the half-velocity point ($r = r_{1/2}$) at the nozzle exit plane ($x = 0$).

To the lowest order, the outer solution may be expressed in the form of an inverse Fourier transform.

$$p_{\text{outer}}(r, x, \phi, t; n, \omega) = \int_{-\infty}^{\infty} \hat{A}_n(\omega)\hat{g}_n(\eta, \omega)H_n^{(1)}[i\lambda(\eta, \omega)r]$$

$$\exp\,[i(\eta x + n\phi - \omega t + \pi/2)]\; d\eta \tag{5}$$

$$\hat{g}_n(\eta, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\,\{i\,[\theta_n(x, \omega) - \eta x] - \beta_n(x, \omega)\}\; dx \tag{6}$$

The nonparallel instability wave solution with angular frequency $-\omega$ and mode number $-n$ is related to the solution with positive frequency ω and mode number n (see Refs. 4–6). For eigensolution subjected to normalization (2), it is easy to show that the following relationships are valid.

$$\begin{aligned} \theta_{-n}(x, -\omega) &= -\theta_n^*(x, \omega) \\ \beta_{-n}(x, -\omega) &= \beta_n^*(x, \omega) \\ \hat{p}_{-n}(r, x, -\omega) &= \hat{p}_n^*(r, x, \omega) \\ H_{-n}^{(1)}[i\lambda(x, -\omega)r] &= H_n^{(1)*}[i\lambda(x, \omega)]\,e^{i\pi} \end{aligned} \tag{7}$$

where the asterisk denotes the complex conjugate. By applying Eqs. (7) to Eq. (6) it is found that

$$\hat{g}_{-n}(-\eta, -\omega) = \hat{g}_n^*(\eta, \omega) \tag{8}$$

Now near the nozzle exit of a high Reynolds number supersonic jet, the turbulent flow in the mixing layer has no intrinsic length and time scales. As a result, the mean velocity profile as well as the turbulence statistics exhibit self-similarity. The lack of intrinsic length and time scales suggests that the spectrum of instability waves of the jet may be regarded as having been initiated by random excitation having no characteristic length and time scales. In other words, they are excited by white noise at the nozzle exit. In recent works on broadband shock associated noise, Tam[14,15] used this reasoning to formulate a stochastic model of the instability wave spectrum for shock noise prediction. However, for practical application to the broadband shock associated noise problem the white noise spectrum was not actually used. Instead a simpler self-similarity spectrum was adopted. Prior to the work of Tam,[14,15] Tam and Chen[4] and Plaschko[5,6] had developed a similar stochastic model to predict the characteristics of turbulence in two-dimensional mixing layers and jets with very encouraging results. Here, such a stochastic instability wave model is adopted for predicting the dominant part of the turbulent mixing noise of supersonic jets from first principle.

For applications to jets from aircraft engines, the self-similarity assumption may become less accurate. In full-scale engines, the intensity of core turbulence, swirl, and exit profile nonuniformities could be substantial. When this is true, slight modifications to the model may be required.

Since the instability wave spectrum is assumed to be initiated by white noise excitation, it is equivalent to prescribing the condition that the pressure autocorrelation function at the half-velocity point at the nozzle exit has the form of a product of delta functions. That is, if $p(r, x, \phi, t)$ is the total pressure fluctuation associated with the instability waves, then

$$\langle p(r_{1/2}, 0, \phi, t)p(r_{1/2}, 0, \phi + \Phi, t + \tau)\rangle = 2\pi^2\tilde{D}\delta(\Phi)\delta(\tau) \tag{9}$$

where $\langle\ \rangle$ denotes ensemble average, and $\delta(\ )$ is the Dirac delta function. $\tilde{D}$ is a constant representing the area of the autocorrelation function. It is the only unknown of the present theory.

The total pressure fluctuation $p(r, x, \phi, t)$ can be found by summing up all of the contributions from instability waves of all frequencies and azimuthal mode numbers. By means of Eqs. (1), (3), and (4) it is found inside the jet

$$p(r, x, \phi, t) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} a_n(\omega)\frac{\hat{p}_n(r, x, \omega)}{|\hat{p}_n(r_{1/2}, 0, \omega)|}$$

$$\cdot \exp\left[i\left(\theta_n + n\phi - \omega t + \frac{\pi}{2}\right) - \beta_n\right] d\omega \tag{10}$$

By substituting Eq. (10) into Eq. (9) an integral equation, by which the ensemble average of the stochastic random function $a_n(\omega)$, $\langle a_n(\omega)a_{n'}(\omega')\rangle$ may be determined, is obtained.

$$\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\iint_{-\infty}^{\infty} \langle a_n(\omega)a_{n'}(\omega')\rangle\hat{p}_n(r_{1/2}, 0, \omega)$$

$$\cdot\frac{\hat{p}_{n'}(r_{1/2}, 0, \omega')\exp\{i\,[(n + n')\phi - (\omega + \omega')t + n\Phi - \omega\tau + \pi]\}}{|\hat{p}_n(r_{1/2}, 0, \omega)||\hat{p}_{n'}(r_{1/2}, 0, \omega')|}$$

$$d\omega\, d\omega' = 2\pi^2\tilde{D}\delta(\Phi)\,\delta(\tau) \tag{11}$$

It can readily be shown, by direct substitution and the use of relations (7) that the solution of Eq. (11) is

$$\langle a_n(\omega)a_{n'}(\omega')\rangle = (\tilde{D}/2)\delta(\omega + \omega')\delta_{n,-n'} \tag{12}$$

where $\delta_{n,m}$ is the Kronecker delta.

Now in the far field, the pressure fluctuation associated with the spectrum of instability waves is given by summing up all of the contributions from waves of all frequencies and azimuthal mode numbers. On using Eqs. (5) and (4) we have

$$p(r, x, \phi, t) = \sum_{n=-\infty}^{\infty}\iint_{-\infty}^{\infty}\frac{a_n(\omega)}{|\hat{p}_n(r_{1/2}, 0, \omega)|}\hat{g}_n(\eta, \omega)$$

$$\cdot H_n^{(1)}[i\lambda(\eta, \omega)r]\exp\,[i(\eta x + n\phi - \omega t + \pi/2)]\, d\eta\, d\omega \tag{13}$$

The pressure autocorrelation function at a point (r, x, ϕ) in the far field is

$$\langle p(r, x, \phi, t)p(r, x, \phi, t + \tau)\rangle = \sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\iiint_{-\infty}^{\infty}\!\!\int$$

$$\frac{\langle a_n(\omega)\,a_{n'}(\omega')\rangle}{|\hat{p}_n(r_{1/2}, 0, \omega)\hat{p}_{n'}(r_{1/2}, 0, \omega')|}$$

$$\cdot\,\hat{g}_n(\eta, \omega)\hat{g}_{n'}(\eta', \omega')H_n^{(1)}[i\lambda(\eta, \omega)\,r]H_{n'}^{(1)}[i\lambda(\eta', \omega')r]$$

$$\cdot\exp\{i\,[(\eta + \eta')x + (n + n')\phi - (\omega + \omega')t - \omega\tau + \pi]\}$$

$$d\eta\, d\eta'\, d\omega\, d\omega' \tag{14}$$

On using Eq. (12) and relations (7) the right side of Eq. (14) can be greatly simplified. This gives

$$\langle p(r, x, \phi, t) p(r, x, \phi, t+\tau)\rangle = \frac{\tilde{D}}{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|G_n(r, x, \omega)|^2}{|\hat{p}_n(r_{1/2}, 0, \omega)|^2} e^{-i\omega\tau}\, d\omega \tag{15}$$

where

$$G_n(r, x, \omega) = \int_{-\infty}^{\infty} \hat{g}_n(\eta, \omega) H_n^{(1)}[i\lambda(\eta, \omega) r] e^{i\eta x}\, d\eta \tag{16}$$

The power spectrum $S(r, x, \phi, \omega)$ is the Fourier transform of the autocorrelation function, i.e.,

$$S(r, x, \phi, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle p(r, x, \phi, t) p(r, x, \phi, t+\tau)\rangle e^{-i\omega\tau}\, d\tau \tag{17}$$

By means of Eq. (15) and on integrating over τ and then ω, the following noise power spectrum formula is found.

$$S(r, x, \phi, \omega) = \frac{\tilde{D}}{2} \sum_{n=-\infty}^{\infty} \frac{|G_n(r, x, \omega)|^2}{|\hat{p}_n(r_{1/2}, 0, \omega)|^2} \tag{18}$$

Equation (18) can be further simplified by evaluating the integral of Eq. (16) asymptotically. Let (R, θ, ϕ) be the coordinates of a spherical coordinate system centered at the nozzle exit. (R, θ, ϕ) are related to the cylindrical coordinate system (r, ϕ, x) by

$$x = R\cos\theta, \qquad r = R\sin\theta$$

and ϕ is the same. For large R, Eq. (16) can be evaluated by the method of stationary phase so that Eq. (18) may be rewritten in the form

$$\tilde{S}(R, \theta, \phi, f) = 10 \log_{10} \left[\frac{\rho_j^2 u_j^3 R_j^3}{p_{\mathrm{ref}}^2 R^2} \sum_{n=-\infty}^{\infty} \frac{8\pi\tilde{D}|\hat{g}_n(\rho_\infty^{1/2} M_j \omega\cos\theta, \omega)|^2}{|\hat{p}_n(r_{1/2}, 0, \omega)|^2} \right] \tag{19}$$

where $\tilde{S}$ is the noise power spectrum in decibels (re: per cycle per unit time), $f(\omega = 2\pi f)$ is the frequency, and $p_{ref} = 2 \times 10^{-5}$ N/m^2 is the reference pressure for the decibel scale. In Eq. (19), the quantities inside the summation are dimensionless whereas those outside are dimensional.

Similarly, on proceeding as earlier the (dimensionless) mean squared axial velocity fluctuation associated with the instability wave spectrum at the half-velocity point at the nozzle exit plane is found to be given by

$$\langle u^2(r_{1/2}, 0, \phi, t)\rangle = \frac{\tilde{D}}{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{u}_n(r_{1/2}, 0, \omega)|^2}{|\hat{p}_n(r_{1/2}, 0, \omega)|^2}\, d\omega \tag{20}$$

where $\hat{u}_n(r, x, \omega)$ is the axial velocity instability wave eigenfunction.

III. Numerical Results and Comparison with Experiments

High quality supersonic jet noise data were not readily available in the literature until the recent work of Seiner et al.[3] As far as is known, this is the only study in which narrow-band (122-Hz bandwidth) jet noise data at very high jet temperature and velocities were measured. In the experiments, a convergent-divergent nozzle of Mach 2 design was used. By means of a propane burner of advanced design the total temperature of the jet T_0 was able to reach 1534 K. Microphones spaced at approximately 3-deg intervals were used. This arrangement allowed the measurement of highly accurate noise directivity data. In this section the data of Seiner et al.[3] will be used to test the numerical results of the stochastic instability wave model theory. A careful examination of the measured data reveals that there are distinct differences between the low temperature jet noise data and those at higher temperatures (although the jet Mach number is the same). For jets with (static) temperature equal to or less than ambient temperature the dominant part of the jet noise (see Fig. 1) is radiated primarily to a relatively narrow angular sector in the downstream direction. In addition, the directivity curves at fixed Strouhal number are very smooth. In contrast, for jets at much higher temperatures the dominant part of jet noise is radiated to a much wider angular sector. The directivity curves, especially those at large Strouhal numbers, are highly irregular and have multiple peaks. To ensure a rigorous test of the theory we have selected the case of isothermal jet (T_0 = 500 K) and the case with T_0 = 1114 K (representative of high temperature jets) for detailed comparisons with numerical results.

It is well known that the characteristics of the instability waves of free shear flows and jets are greatly affected by the mean velocity and density profiles. For this reason, it is desirable to use smooth profile functions which match the measured mean flow parameters. In this work, the following error function profile with three parameters is used to represent the mean axial velocity $\bar{u}(r, x)$ of the jet. This profile fits the Mach 2 jet radial mean velocity distributions measured by Seiner et al.[16] quite well.

$$\frac{\bar{u}(r, x)}{u_c} = \frac{1}{2}\left\{1 - \mathrm{erf}\left[\ln 2\left(\frac{r}{r_{1/2}} - \left(\frac{r_{1/2}}{r}\right)^{3/4}\right)\frac{r_{1/2}}{b}\right]\right\} \tag{21}$$

In Eq. (21), erf [ ] is the error function. The parameters u_c, $r_{1/2}$, and b are functions of x where u_c is the centerline velocity of the jet, $r_{1/2}$ is the radial distance to the half-velocity point, and b is the half-width of the jet mixing layer. In the initial core region of the jet, u_c is equal to u_j, and $b/r_{1/2}$ is very small. In the fully developed region of the jet farther downstream, u_c/u_j becomes smaller and smaller whereas b is equal to $r_{1/2}$. Figure 2 shows the centerline velocity distribution used in the present calculations. The isothermal jet has a much longer potential core. Also shown are the measured values of u_c obtained by Seiner et al.[3] Figure 3 shows the

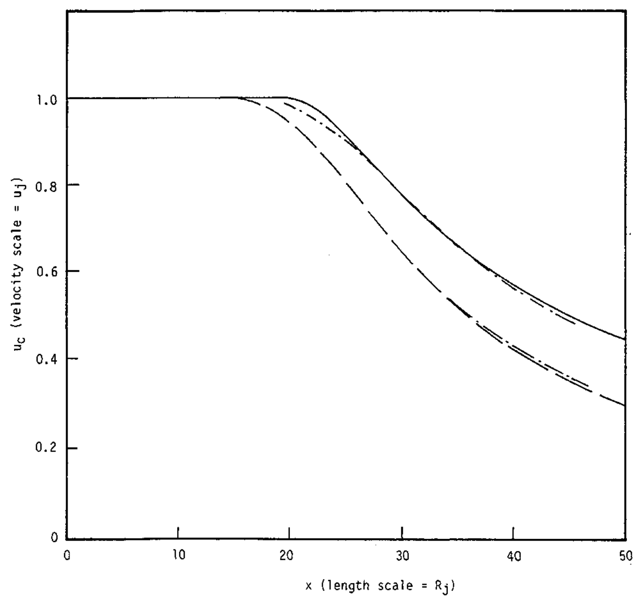

Fig. 2 Comparisons between the jet centerline velocity distributions used in the stochastic wave model calculations and experimental data, M_j = 2: model —— T_0 = 500 K, – – – T_0 = 1114 K, and — · — · — measurements.[3]

axial distributions of b and $r_{1/2}$ used in the present calculation for the $T_0 = 500$ K jet. The initial mixing layer thickness is $b(0)/R_j = 0.04$. Figure 4 shows those for the $T_0 = 1114$ K jet with $b(0)/R_j = 0.04$. These values are to be compared with the measured data plotted in the same figure. Clearly from Figs. 2 and 3, the empirical distributions of u_c, $r_{1/2}$, and b of the present calculations fit reasonably well with measurements. In Fig. 4, the $r_{1/2}$ curve has been chosen to agree reasonably well with the measurements. In the analytical model the value of b is determined by the requirement of conservation of momentum flux of the jet. It appears to differ from the experimental data. Presently, we have no explanation for the disagreement. Overall, we believe that Eq. (21) is a satisfactory mean velocity profile for testing the stochastic instability wave model theory.

Seiner et al.[3] did not provide any mean temperature or jet density measurements. In the absence of experimental data, the Crocco's relation is used to calculate the mean density distribution across any cross section of the jet. The explicit relation is

$$\frac{\rho_j}{\bar{\rho}} = \left(1+\frac{\gamma-1}{2}M_j^2\right)\left[\frac{T_\infty}{T_0}+\left(1-\frac{T_\infty}{T_0}\right)\frac{\bar{u}}{u_j}\right]-\frac{\gamma-1}{2}M_j^2\left(\frac{\bar{u}}{u_j}\right)^2 \quad (22)$$

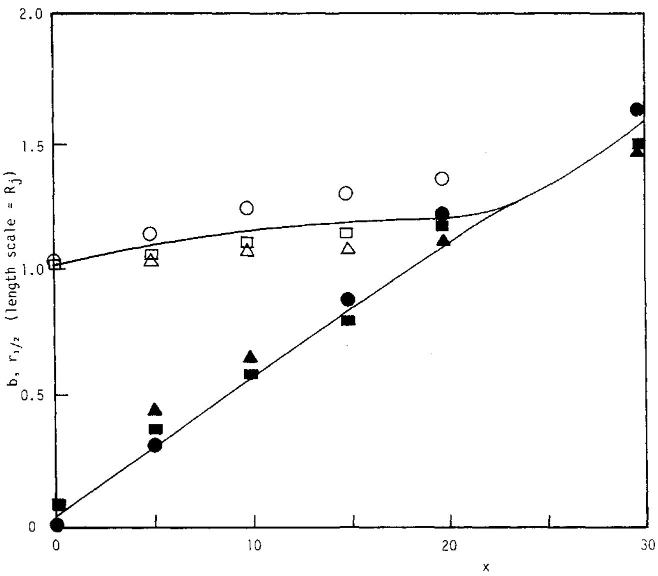

Fig. 3 Comparison between the axial distribution of b and $r_{1/2}$ used in the stochastic wave model calculation and measurements[3]: model ——, $T_0 = 500$ K; experiment: ○, □, △; $r_{1/2}$ at 313 K, 755 K, and 1114 K; ●, ■, ▲ b at 313 K, 755 K, 1114 K.

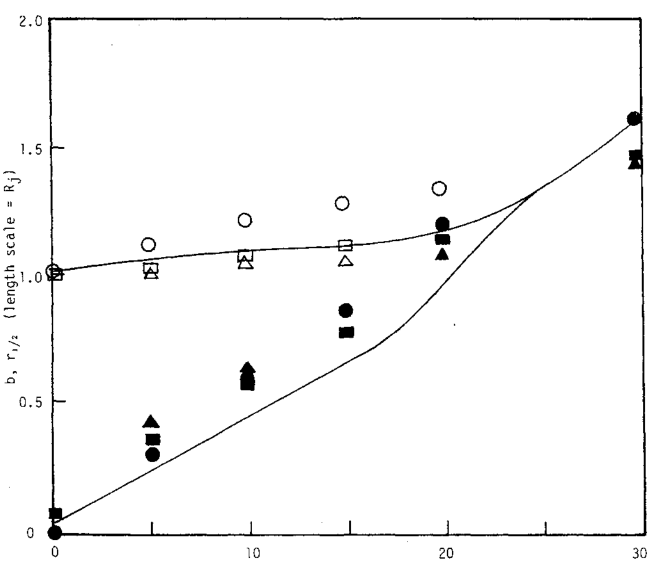

Fig. 4 See legend of Fig. 3, model ——, $T_0 = 1114$ K.

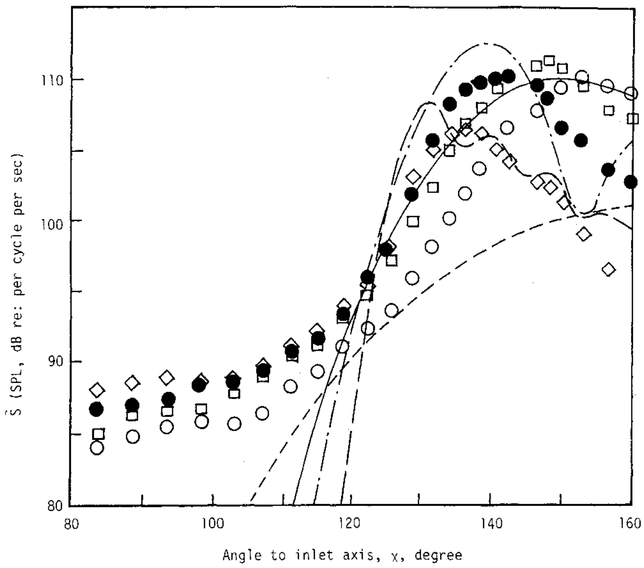

Fig. 5 Comparisons between calculated and measured noise directivities, $M_j = 2.0$, $T_0 = 500$ K: 0.067 St, ------ calculated, ○ measured[3]; 0.12 St, —— calculated, □ measured[3]; 0.20 St, – · – · – calculated, ● measured[3]; and 0.40 St, – – – calculated, ◊ measured.[3]

where T_∞ is the ambient temperature and γ is the specific heat ratio of the gas. Equation (22) was also used in Ref. 11.

In the single instability wave theory of Ref. 1, the use of a multiple-scales expansion implicitly assumes that the nonparallel effect is small. For jet flows, it is known that the spreading rate decreases with Mach number. Thus, for highly supersonic jets the nonparallel flow correction should be negligible. To compute the nonparallel flow correction factor β_n of Eq. (3), values of the derivatives of the mean flow and eigenfunction in the axial direction are required. In the absence of exact jet mean flow and density profiles accurate determination of these derivatives becomes fairly difficult. Since the main purpose here is to test the validity of the stochastic instability wave model theory, for simplicity, the small nonparallel flow effect has been neglected (i.e., β_n is set equal to zero) in all of the numerical results to be reported subsequently.

It is well known that if the inviscid theory (Rayleigh equation) is used to calculate the characteristics of the instability waves in the developed region of the jet where the waves are damped, the calculation cannot be carried out in the physical plane along the real r axis. The computation can only be done by the deformed contour integration method,[7] which requires analytic continuation of the solution into the complex r plane. For weakly damped waves, this procedure is feasible and efficient. However, if the wave is moderately damped, the results calculated by the deformed contour integration method is not always reliable unless an exact mean flow profile is known. To be able to integrate the Rayleigh equation over the deformed contour in the complex r plane, it is necessary to have the analytic continuation of the mean velocity profile off the real r axis. If only an empirically fitted velocity profile is available, there is no known way to perform analytic continuation accurately. To avoid this type of uncertainly and possible error, we have incorporated turbulent (eddy) viscosity terms in the instability wave equations (the compressible Orr-Sommerfeld equations). With turbulent viscosity terms added, the integration can be carried out completely along the real r axis. Physically, the turbulent viscosity terms may be regarded as simulating the effect of fine-scale turbulence on the large turbulence structures/instability waves. In the present work, a local turbulent Reynolds number $R_t = u_j b/\nu_t$ (where ν_t is the kinematic turbulent viscosity and b the half-width of the jet mixing layer) of 500 is used for the first instability wave calculations. The calculations are then repeated with $R_t = 1000, 2000$, up to 5000. It is found that the computed instability wave characteristics are nearly independent of turbulent Reynolds number for $R_t \geq 2000$. In other words, the viscous (eddy) instabil-

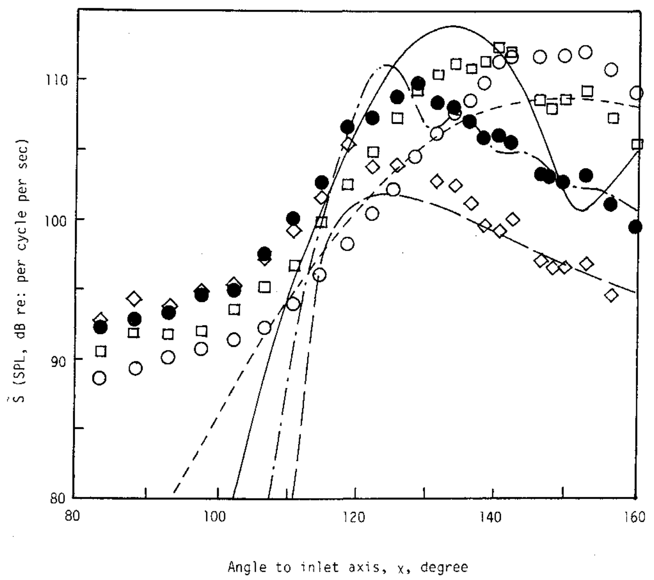

Fig. 6 Comparisons between calculated and measured noise directivities, $M_j = 2.0$, $T_0 = 1114$ K: 0.056 St, ------ calculated, ○ measured[3]; 0.10 St, —— calculated, □ measured[3]; 0.20 St, – · – · – calculated, ● measured[3]; and 0.40 St, ––– calculated, ◊ measured.[3]

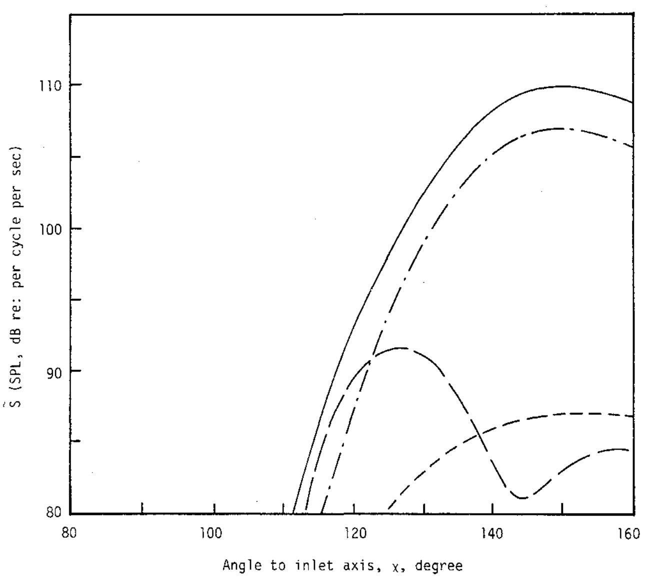

Fig. 7 Relative contributions to the radiated noise from different azimuthal instability wave modes, $M_j = 2.0$, $T_0 = 500$ K, $St = 0.12$: ––– $n = 0$; – · – · – $n = \pm 1$; ------ $n = \pm 2$ mode; —— total.

ity results have nearly reached their asymptotic values ($R_t \to \infty$) at approximately $R_t = 2000$. With this understanding, we regard $R_t = 2000$ as the inviscid limit. Accordingly, all of the numerical results reported next are calculated at a turbulent Reynolds number of 2000.

Figure 5 shows the computed noise directivities at Strouhal numbers 0.067, 0.12, 0.2, and 0.4 for the Mach 2.0 jet at a total temperature of 500 K (isothermal jet). A preliminary version of these results has been presented earlier.[17] The unknown constant $\tilde{D}$ of the stochastic wave model theory has been set equal to 4.35×10^{-10} for best overall comparison with experiment. The measurements of Seiner et al.[3] are also plotted in this figure. On comparing the measured and calculated directivities it is clear that there is good agreement for the three higher Strouhal numbers. Except for the very small exhaust angles measured from the jet axis at Strouhal number 0.2, the difference between the calculated and the measured directivities is no more than 2.5 dB. The calculated sound pressure level at Strouhal number 0.067 is, however, low compared with measurements. The reason for this discrepancy is unknown at this time.

Figure 6 shows the computed directivities at Strouhal number 0.056, 0.1, 0.2, and 0.4 for the Mach 2.0 jet at 111 K total temperature and the measured data. The value of $\tilde{D}$ has been chosen to be 9.19×10^{-10} for the computation. In this case, it appears that there is overall good agreement between theory and measurements for all Strouhal numbers. The discrepancy is no more than 2.5 dB except for Strouhal number 0.1 at very low exhaust angles. The peak directions are correctly predicted. The half-widths of the directivity curves also seem to match the data well. On considering that only one parameter, namely $\tilde{D}$, can be adjusted, the overall reasonable agreement between measurements and numerical results may be viewed as strongly suggesting that the theory must contain the essential physics of supersonic jet noise generation.

Within the framework of the stochastic instability wave model theory all of the azimuthal wave modes contribute to the turbulent mixing noise of the jet. Let us now examine the changes in the contributions of different modes as the jet temperature increases. It turns out at low jet temperature the $n = \pm 1$ (helical or flapping) modes are the dominant source of noise. Figure 7 gives the individual contribution of the $n = 0, \pm 1$, and ± 2 instability wave modes to the radiated noise at Strouhal number 0.12 for the Mach 2 jet at 500 K total temperature. Since essentially only the $n = \pm 1$ waves contribute to the radiated noise, the directivity curve would, therefore, be smooth and consist of a single peak. For high temperature jets the situation is different particularly at high Strouhal numbers. The $n = \pm 1$ modes are no longer totally dominant. The higher order modes become increasingly important. Figure 8 gives the individual contributions of the $n = 0, \pm 1, \pm 2, \pm 3, \pm 4$, and ± 5 instability wave modes to the total noise of the jet at Strouhal number 0.4 for the Mach 2.0 jet at 1114 K total temperature. Clearly, no single wave mode is the dominant source of noise. Even the fifth azimuthal mode contributes a small amount of noise radiation in the small exhaust angle directions. The contribution of the sixth azimuthal mode is, however, negligibly small. The fact that more than one mode contributes significantly to the total jet noise causes the noise directivity curve to become fairly broad. This is in total agreement with experimental measurements. In a recent work, Seiner et al.[18] also arrived at a similar conclusion based on single instability wave calculations.

Finally, Eq. (20) permits us to estimate the root-mean-squared axial velocity fluctuation associated with the large turbulence structures/instability waves in the jet mixing layer at the nozzle exit. By using the values of $\tilde{D}$ already determined, we find $\langle \tilde{u}^2(r_{1/2}, 0, \phi, t)\rangle^{1/2}$ to be equal to 0.015 for the $T_0 = 500$ K jet and 0.012 for the $T_0 = 1114$ K jet. At the present time, no experi-

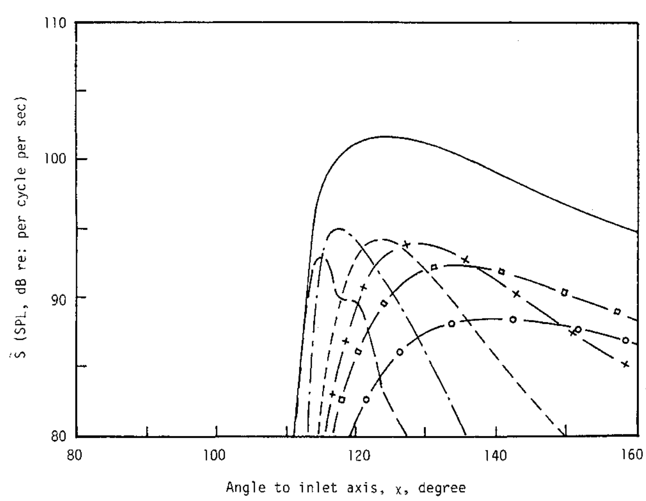

Fig. 8 Relative contributions to the radiated noise from different azimuthal instability wave modes, $M_j = 2.0$, $T_0 = 1114$ K, $St = 0.4$: ––– $n = 0$; – · – · – $n = \pm 1$;------ $n = \pm 2$; – + – + – $n = \pm 3$; –□–□– $n = \pm 4$; – ◦ – ◦ – $n = \pm 5$; —— total.

mental data of the turbulence intensity of high Mach number jets, especially at high temperatures, is available in the literature. However, for lower Mach number jets, turbulence intensity of 1.5% at the nozzle exit region is not uncommon. Here, we believe that the axial turbulent velocities required by the theory are not unreasonable.

IV. Summary and Discussion

In this work, a stochastic instability wave model theory of the large turbulence structures and turbulent mixing noise of supersonic jets has been developed. The theory is an extension of the single instability wave solution of Tam and Burton.[1] In the theory, the large turbulence structures in the mixing layer of the jet are modeled by a linear superposition of instability waves of all frequencies and azimuthal mode numbers. The initial amplitudes (near the nozzle exit) of the instability waves are regarded as stochastic random functions, reflecting on the stochastic nature of the large turbulence structures. The statistical property of the random amplitude functions is determined by the requirement that the instability wave spectrum at the nozzle exit has no intrinsic characteristic length and time scales. The theory can predict the spectrum and directivity of the dominant part of the turbulent mixing noise of supersonic jets and the turbulence intensity associated with the large turbulence structures up to a single multiplicative constant. Numerical results of the theory appear to compare favorably with experimental measurements.

It must be pointed out at this time that at lower jet Mach numbers and temperatures the propagation speeds of the instability waves are greatly reduced. The Mach wave noise generation mechanism, in turn, would become less efficient. For such jets, the contribution of noise from the fine-scale turbulence cannot be ignored. A comprehensive theory of turbulent jet mixing noise which accounts for both noise components has yet to be developed.

Acknowledgments

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